HW 7 : p140 : 7, 17 . p148 : 4, 11

p.140 Q7,

Let $f(x) := x - \cos x$

$f(0) = -1$, $f(\frac{\pi}{2}) = \frac{\pi}{2}$

$f(\frac{\pi}{4}) = 0.07829\ldots > 0$, $\frac{\pi}{4} = 0.785398\ldots$

$f(\frac{\pi}{8}) = -0.531180\ldots < 0$, $\frac{\pi}{8} = 0.39269908\ldots$

$f(\frac{1}{2}(\frac{\pi}{4}+\frac{\pi}{8})) = -0.2424209\ldots < 0$, $\frac{1}{2}(\frac{\pi}{4}+\frac{\pi}{8}) = 0.5890486\ldots = \frac{3\pi}{16}$

$f(\frac{1}{2}(\frac{\pi}{4}+\frac{3\pi}{16})) = -0.0857870\ldots < 0$, $\frac{1}{2}(\frac{\pi}{4}+\frac{3\pi}{16}) = 0.68722339\ldots = \frac{7\pi}{32}$

$f(\frac{1}{2}(\frac{7\pi}{32}+\frac{\pi}{4})) = -0.004640\ldots < 0$, $\frac{1}{2}(\frac{7\pi}{32}+\frac{\pi}{4}) = \frac{15\pi}{64} = 0.736310\ldots$

$f(\frac{1}{2}(\frac{15\pi}{64}+\frac{\pi}{4})) = 0.036607\ldots > 0$, $\frac{1}{2}(\frac{15\pi}{64}+\frac{\pi}{4}) = \frac{31\pi}{128} = 0.7608544\ldots$

$f(\frac{1}{2}(\frac{31\pi}{128}+\frac{15\pi}{64})) = 0.01592835\ldots > 0$, $\frac{1}{2}(\frac{31\pi}{128}+\frac{15\pi}{64}) = \frac{61\pi}{256} = 0.7485826\ldots$

$f(\frac{1}{2}(\frac{61\pi}{256}+\frac{15\pi}{64})) = 0.005630132\ldots > 0$, $\frac{1}{2}(\frac{61\pi}{256}+\frac{15\pi}{64}) = \frac{121\pi}{512} = 0.742446\ldots$

$\longrightarrow f(\frac{1}{2}(\frac{121\pi}{512}+\frac{15\pi}{64})) = 0.00049141\ldots > 0$, $\frac{1}{2}(\frac{121\pi}{512}+\frac{15\pi}{64}) = \frac{241\pi}{1024} = 0.7393787\ldots$

$f(\frac{1}{2}(\frac{241\pi}{1024}+\frac{15\pi}{64})) = -0.002075\ldots < 0$, $\frac{1}{2}(\frac{241\pi}{1024}+\frac{15\pi}{64}) = \frac{481\pi}{2048} = 0.737844\ldots$

$\longrightarrow f(\frac{1}{2}(\frac{481\pi}{2048}+\frac{241\pi}{1024})) = -0.000792\ldots < 0$, $\frac{1}{2}(\frac{481\pi}{2048}+\frac{241\pi}{1024}) = \frac{963\pi}{4096} = 0.7386117\ldots$

$f(\frac{1}{2}(\frac{963\pi}{4096}+\frac{241\pi}{1024})) = -0.000150\ldots < 0$, $\frac{1}{2}(\frac{963\pi}{4096}+\frac{241\pi}{1024}) = \frac{1927\pi}{8192} = 0.7389952\ldots$

We note the two arrows, the solution c satisfies $\underset{\frac{963\pi}{4096}}{\overset{\shortparallel}{0.7386117\ldots}} < c < 0.7393787\ldots = \frac{241\pi}{1024}$

Where $\left|\frac{963\pi}{4096} - \frac{241\pi}{1024}\right| < 0.001$ $\therefore$ Any number between $\frac{963\pi}{4096}$ and $\frac{241\pi}{1024}$ would be "good" approximation. Take $\frac{1927\pi}{8192}$ for instance.

P. 140 Q17, Yes. $f([0,1]) = [m, M]$ for some $m \le M$

if $m < M$, then both $\mathbb{Q} \cap (m,M)$ and $\mathbb{R}\setminus\mathbb{Q} \cap (m,M) \neq \phi$ $\therefore$ $m = M$

and f must be constant.

P. 148 Q4, Note $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to-\infty} f(x)$, f must be uniformly continuous.

See "Tutorial (Nov 24) $\longrightarrow$ Exam (2015-2016) 4(b)" in our webpage

OR :

$$\left|\frac{1}{1+x^2} - \frac{1}{1+y^2}\right| = \left|\frac{y^2-x^2}{(1+x^2)(1+y^2)}\right| = |x-y|\left(\frac{|x+y|}{(1+x^2)(1+y^2)}\right) \le |x-y|\left(\frac{|x|+|y|}{1+x^2+y^2}\right)$$

$$= |x-y|\left(\frac{|x|+|y|}{(x^2+\frac{1}{2})+(y^2+\frac{1}{2})}\right)$$

We claim that $\frac{|x|+|y|}{(x^2+\frac{1}{2})+(y^2+\frac{1}{2})} \le 1$ $\forall x, y$. To see that $\frac{|x|}{(x^2+\frac{1}{2})} \le 1 \iff 2|x| \le 2x^2+1 \iff 0 \le 2(|x|-\frac{1}{2})^2+\frac{1}{2}$

P.148 Q4 (Cont'd): $\therefore \dfrac{|x|+|y|}{(x^2+\frac{1}{2})+(y^2+\frac{1}{2})} \leq 1 \quad \forall\, x,y \in \mathbb{R}$

$\Rightarrow f(x) = \dfrac{1}{1+x^2}$ is Lipschitz, hence uniformly continuous.

P.148 Q11, g is continuous on $[0,1]$ $\therefore$ uniformly continuous.

Suppose $\exists$ constant K s.t. $|g(x)| \leq K|x| \quad \forall\, x \in [0,1]$

We see that $\dfrac{|g(x)|}{|x|} \leq K \quad \forall\, x \in (0,1]$

Now, $\dfrac{|g(x)|}{|x|} = \dfrac{1}{\sqrt{x}} \to \infty$ as $x \to 0^+$ Contradicts to

$\left|\dfrac{g(x)}{x}\right| \leq K \quad \forall\, x \in (0,1]$

$\therefore \nexists$ constant K s.t. $|g(x)| \leq K|x| \quad \forall\, x \in [0,1]$.

It is not Lipschitz: Suppose it is Lipschitz, $\exists\, K$ s.t.

$|g(x) - g(y)| \leq K|x-y| \quad \forall\, x,y \in [0,1]$

In particular, take $y = 0$, $|g(x)| \leq K|x| \quad \forall\, x \in [0,1]$

Contradiction.

HW8 p.148: 2, 3, 6, 7.

Q2. Let $x,y \in [1,\infty)$, assume $y > x$.

$|f(x) - f(y)| = \left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2-x^2}{x^2y^2}\right| = |y-x|\frac{(x+y)}{x^2y^2}$

$\leq |y-x|\dfrac{2y}{x^2y^2} \leq 2|y-x|$

$\therefore f$ is Lipschitz on $[1,\infty)$, hence uniformly cont.

f is not uniformly continuous on $B := (0,\infty)$:

Suppose f is uniformly continuous, By Theorem 5.4.7, $\{f(\frac{1}{n})\}_{n\geq 1}^{\infty}$ is Cauchy

But, $f(\frac{1}{n}) = n^2 \to \infty$ as $n \to \infty$ Contradiction.

Q3. (a) We use (iii), take $x_n = n$, $u_n = n + \frac{1}{n}$, $|x_n - u_n| = \frac{1}{n} \to 0$ as $n \to \infty$

but $|f(x_n) - f(u_n)| = 2 + \frac{1}{n^2} \geq 2$

(b) We use (iii), take $x_n = \dfrac{1}{2n\pi}$, $u_n = \dfrac{1}{2n\pi + \frac{\pi}{2}}$, $|x_n - u_n| = \dfrac{\frac{\pi}{2}}{(2n\pi)(2n\pi+\frac{\pi}{2})} \leq \dfrac{1}{2n\pi}$ for $n \geq 1$

$\therefore |x_n - u_n| \to 0$ as $n \to \infty$,

but $|g(x_n) - g(u_n)| = 1$

HW8 :
P.148

Q6 , Assume $|f(x)| \le M_1$, $|g(x)| \le M_2$ $\forall x \in A$ for some $M_1, M_2 > 0$ (by boundedness of f, g)

Let $\varepsilon > 0$

Since both f, g are uniformly continuous on A ,

$\exists \delta_1, \delta_2 > 0$ s.t.

$$|f(x)-f(y)| < \frac{\varepsilon}{M_1+M_2} \quad \forall x, y \in A \text{ w/ } |x-y| < \delta_1$$

$$|g(x)-g(y)| < \frac{\varepsilon}{M_1+M_2} \quad \forall x, y \in A \text{ w/ } |x-y| < \delta_2$$

Let $\delta := \min\{\delta_1, \delta_2\}$. then $\forall x, y \in A$ w/ $|x-y| < \delta$, we have

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &\le |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)| \\ &\le |f(x)|\,|g(x)-g(y)| + |g(y)|\,|f(x)-f(y)| \\ &\le M_1 |g(x)-g(y)| + M_2 |f(x)-f(y)| \\ &< M_1 \cdot \frac{\varepsilon}{M_1+M_2} + M_2 \frac{\varepsilon}{M_1+M_2} = \varepsilon \end{aligned}$$

Q7. f is Lipschitz.

g is also Lipschitz because $|g(x)-g(y)| = |\sin x - \sin y| = |\cos \xi|\,|x-y| \le |x-y|$ for some ξ between x, y by Mean value theorem.

Otherwise, Observe g is periodic and go back to " Tutorial (Nov 24.)

$\longrightarrow$ Extra Tutorial 2 (2015-2016) Q5 " in our webpage . Hence g is uniformly continuous.

fg is not uniformly continuous on $\mathbb{R}$: We use (iii) of Criterion 5.4.2.

Take $x_n = 2n\pi$, $u_n = 2n\pi + \frac{1}{n}$, then $|x_n - u_n| \to 0$ as $n \to \infty$

But $|f(x_n)g(x_n) - f(u_n)g(u_n)| = |0 - (2n\pi+\frac{1}{n})\sin(2n\pi+\frac{1}{n})|$

$= (2n\pi+\frac{1}{n})\sin\frac{1}{n}$

Now , we use " $\lim_{x\to 0} \frac{\sin x}{x} = 1$ " : $\exists \delta > 0$ s.t. $\left|\frac{\sin x}{x} - 1\right| < \frac{1}{2}$ $\forall x \in (-\delta, \delta) \setminus \{0\}$

Hence $\frac{1}{2} < \frac{\sin x}{x} < \frac{3}{2}$ $\forall x \in (-\delta,\delta)\setminus\{0\}$. By AP $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \delta$

$\forall n \ge N$, we have $\sin\frac{1}{n} > \frac{1}{2}(\frac{1}{n})$ $\therefore$ $|f(x_n)g(x_n) - f(u_n)g(u_n)| > \pi$ $\forall n \ge N$